$1/f$ noise and very high spectral rigidity

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It was recently pointed out that the spectral fluctuations of quantum systems are formally analogous to discrete time series, and therefore their structure can be characterized by the power spectrum of the signal. Moreover, it is found that the power spectrum of chaotic spectra displays a $1/f$ behavior, while that of regular systems follows a $1/f^2$ law. This analogy provides a link between the concepts of spectral rigidity and antipersistence. Trying to get a deeper understanding of this relationship, we have studied the correlation structure of spectra with high spectral rigidity. Using an appropriate family of random Hamiltonians, we increase the spectral rigidity up to hindering completely the spectral fluctuations. Analyzing the long range correlation structure a neat power law $1/f$ has been found for all the spectra, along the whole process. Therefore, $1/f$ noise is the characteristic fingerprint of a transition that, preserving the scale-free correlation structure, hinders completely the fluctuations of the spectrum.

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INTRODUCTION

The statistical study of energy level fluctuations is one of the most important tools for understanding quantum chaos. Among the relevant results of the statistical theory of spectral fluctuations, we note that the level fluctuations of quantum systems whose classical analogues are fully integrable are well described by Poisson statistics, i.e., the successive energy levels are not correlated [1]; on the other hand, the fluctuation properties of generic quantum systems which in the classical limit are fully chaotic coincide with those of the classic random matrix ensemble (RME) [2]. As a consequence of these results, generic quantum systems (with or without a clear classical analogs) are usually said to be integrable or chaotic depending whether the statistical properties of their spectra coincide with those of a noncorrelated sequence or those of the RME. A complete review of these features and later developments can be found in [3,4].

Recently, a new approach to energy level fluctuations, based on traditional methods of time series analysis, has been proposed. Considering the sequence of energy levels as a discrete time series, it has been shown that chaotic quantum systems, characterized by rigid spectra, exhibit $1/f$ noise; whereas integrable quantum systems, with very low spectral rigidity, exhibit $1/f^2$ noise [5,6]. On the other hand, as is well known, time series with $1/f$ noise have maximum antipersistence, whereas time series with $1/f^2$ noise are nonantipersistent. Thus, the formal analogy between time series and energy level spectra provides a link between spectral rigidity and antipersistence.

This result raised the natural question of what happens in intermediate situations between fully regular and fully chaotic motion. The transition between these two types of motion was studied using the Robnik quantum billiard. As the billiard border changes, the transition takes place very smoothly. It was found that the long range correlations of the spectrum are characterized by a $1/f^{\alpha}$ noise through the whole transition, with α smoothly changing from $\alpha=2$ in the regular regime to $\alpha=1$ in the chaotic one. Similar behavior has been found in other intermediate systems [7]. Thus, the quantum chaos framework provides us with a mechanism that starting from a nonantipersistent time series ($1/f^2$ noise) leads to a fully antipersistent time series ($1/f$ noise) by increasing the spectral rigidity of the spectrum. Since the reasons why $1/f$ fluctuations are so ubiquitous are not well understood yet, this result may become very important. Actually, there is a close relationship between quantum chaos and other systems through random matrix theory—as complex zeros of the Riemann function [8] or the traffic flow [9]—and therefore these results might be applied to a broad set of systems.

It is our goal in this paper to get a deeper understanding of the analogy between spectral rigidity and antipersistence. To this purpose we study how the correlation structure of the spectrum changes as the spectral rigidity increases: the maximum level of antipersistence of an energy spectrum is already reached for RME-like spectral fluctuations, but the correlation structure at very high spectral rigidity is unknown. Using the ideas of [10], we introduce a suitable family of RMEs to model the transition from standard chaotic spectra to truly local equidistant spectra, where the spectral rigidity is so high that fluctuations are completely hindered. The main result is that as we further increase spectral rigidity, fluctuations vanish without changing the global correlation structure. In other words, $1/f$ noise is the characteristic fingerprint of a transition that, preserving the scale-free correlation structure, hinders completely the fluctuations of the spectrum.

The paper is organized as follows. The next section gives a rapid survey of concepts like antipersistence, spectral rigidity, and level repulsion. Next we introduce the physical model we use to study the transition from RME-like spectral

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fluctuations to spectra with very high rigidity. Then we explain how the statistical analysis is performed, stressing the relevance of the unfolding procedure, and the main results are presented and discussed. Finally, we give a brief summary and quote the main conclusions.

ANTIPERSISTENCE, SPECTRAL RIGIDITY, AND LEVEL REPULSION

A clear relation between concepts like spectral rigidity, which characterizes quantum chaotic energy level spectra, and antipersistence, which appears in a broad kind of time series, emerges from the formal analogy between energy level spectra and time series.

Rigidity of the energy spectrum means that the deviations of the energy levels from those of a local uniform spectrum are generally small, and that the spectrum is organized in such a way that any deviation of a level from its mean position tends to be balanced by neighboring levels. Therefore, it is unlikely to find long series of consecutive energy levels all above or below their mean position. On the other hand, antipersistence in time series means that an increasing or decreasing trend in the past makes the opposite trend in the future probable. It appears through the whole scale domain of the signal and entails self-similarity in the frequency space. Antipersistent time series have $1/f^{\alpha}$ power spectra, with $1<\alpha<2$: the smaller the value of α, the greater intensity of the antipersistence.

Spectral rigidity is related to another property of quantum chaotic systems named level repulsion, i.e., the fact that two nearby levels repel each other with a certain intensity. The nearest-neighbor spacing distribution $P(s)$ gives the probability that the distance between two consecutive levels, measured in units of the local averaging spacing, lies between s and $s+ds$. It is a widely accepted statistic to study the short range correlations of the spectrum. Usually $P(s) \propto s^{\beta}$ when $s \ll 1$. This means that the probability of finding two neighboring levels at a distance s is proportional to s^{β}, provided that s is small enough. Therefore, the exponent β measures the repulsion between consecutive levels. For generic integrable systems $\beta=0$ (levels behave as noncorrelated random variables). On the other hand, for fully chaotic systems β is universal and depends only on the symmetries of the system: $\beta=1$ for systems with time-reversal invariance and rotational symmetry, or broken rotational symmetry, but integer spin; $\beta=2$ for systems with no time-reversal invariance; and $\beta=4$ for systems with time-reversal invariance, broken rotational symmetry and half-integer spin [11]. As β increases, the spectrum becomes more and more rigid, and the limit $\beta \to \infty$ corresponds to a local equidistant or picket-fence spectrum. The paradigmatic example of this kind of spectrum is the harmonic oscillator or a superposition of harmonic oscillators. These kinds of systems is also integrable, but nongeneric and the results of Berry and Tabor do not apply to them. However, it is of practical interest since the Hamiltonian of many systems in molecular or nuclear physics can be written as an integrable part, producing a picket-fence spectrum plus a chaos inducing part.

The results reported in [5] show that the strong rigidity of chaotic quantum systems with $\beta=1$, 2, and 4 extends through the whole spectrum in a self-similar way [12], giving rise to a $1/f$ noise. On the contrary, regular systems with no level repulsion, i.e., $\beta=0$, lead to nonantipersistent $1/f^2$ noise. Moreover, it has been shown [7] that some intermediate (neither integrable, nor fully chaotic) quantum systems also give rise to antipersistent time series, where α approaches monotonically to unity as the system approaches full chaos. Therefore, we can state that the order-to-chaos transition implies an increase of antipersistence.

TRIDIAGONAL ENSEMBLES

The basic idea is based on the Lanczos tridiagonal reduction of a Hamiltonian matrix [13]. Although the method can be applied to Hermitian complex and quaternionic matrices, we shall only consider real symmetric matrices. Given a Hamiltonian matrix with elements

$$H_{ji}=H_{ij}, \quad i=1,\ldots,N, \quad j=i,\ldots,N, \tag{1}$$

defined in a finite Hilbert space of dimension N, the Lanczos method reduces it to tridiagonal form; therefore, in the new basis the matrix elements satisfy

$$h_{ij}=0, \quad i=1,\ldots,N, \quad |i-j|>1. \tag{2}$$

Moreover, the Lanczos tridiagonal reduction leads to a canonical form for the nonzero matrix elements h_{ii} and h_{ii+1}. They can be separated into smooth and fluctuating parts as follows:

$$h_{ji}=h_{ij}=\bar{h}_{ij}+\tilde{h}_{ij}, \quad i=1,\ldots,N, \quad j=i,i+1. \tag{3}$$

The smooth part $\bar{h}$—which can be viewed as an ensemble average—is given by the four lowest moments of H, and therefore is closely related to the mean level density of the system. The fluctuating part $\tilde{h}$ can be defined as the difference between the actual Hamiltonian and the smooth part, i.e., $\tilde{h}=h-\bar{h}$.

Taking advantage of the canonical tridiagonal form h of any Hamiltonian H, it is possible to define a new family of Hamiltonians $h(F)$

$$h(F)=\bar{h}+F\tilde{h}, \tag{4}$$

where the parameter F is introduced to modulate the amplitude of the matrix elements fluctuations. Let us now suppose that H pertains to a matrix ensemble with joint probability density $P(H)$; then, we can define a set of tridiagonal matrix ensembles depending on the single parameter F. We define the density probability $P(h(F))$ to be equal to that of the original matrix H, i.e.,

$$P(h(F))=P(h(1))=P(H). \tag{5}$$

In what follows we shall call these ensembles random tridiagonal matrix ensembles (RTMEs). Using spin Hamiltonian, nuclear shell-model, and random matrices, all having $\beta=1$ at $F=1$, it was shown [10] that the short range correlations of these ensembles are characterized by the universal law

$$\beta = \frac{1}{F^2}. \tag{6}$$

It appears that the spectral fluctuations are dictated by the amplitude of the matrix elements fluctuations. The repulsion parameter β changes from $\beta=1$ for $F=1$ to $\beta=\infty$ for $F=0$; therefore the spectral fluctuation evolve from those of a chaotic system to those of a picket-fence spectrum.

THE STATISTICAL ANALYSIS

The aim of this work is to study the long-range spectral correlations through the whole transition from $\beta=1$ to the limit $\beta\to\infty$. With this purpose we introduce an appropriate RTME that coincides with the Gaussian orthogonal ensemble (GOE) at $F=1$. The GOE is the paradigm of a chaotic quantum system with time-reversal invariance and rotational symmetry, or broken rotational symmetry, but integer spin.

Unfolding

The first step, previous to any statistical analysis of the spectral fluctuations, is the unfolding of the energy spectrum. For any quantum system, the level density $\rho(E)$ can be separated into a smooth part $\overline{\rho(E)}$ and a fluctuating part $\widetilde{\rho(E)}$; the former gives the main trend of the level density, and the later characterizes the spectral fluctuations. Usually, level fluctuation amplitudes are modulated by the value of the mean level density $\widetilde{\rho}(E)$; therefore, to compare the fluctuations of different systems, the level density smooth behavior must be removed. The unfolding consists in locally mapping the real spectrum into another with mean level density equal to 1. The actual energy levels E_i are mapped into new dimensionless levels ϵ_i,

$$E_i \to \epsilon_i = \bar{N}(E_i), \quad i=1,\ldots,N, \tag{7}$$

where N is the dimension of the spectrum and $\bar{N}(E)$ is given by

$$\bar{N}(E) = \int_{-\infty}^{E} dE' \bar{\rho}(E'). \tag{8}$$

Then, the nearest-neighbor spacing sequence, defined by

$$s_i = \epsilon_{i+1} - \epsilon_i, \quad i=1,\ldots,N-1, \tag{9}$$

satisfies $\langle s\rangle=1$. When F is not very large, we can obtain the smooth part of the level density as $\overline{\rho^{(F)}}(E)=\rho^{(F=0)}(E)$. Thus, $s_i=N^{(F=0)}(E_{i+1})-N^{(F=0)}(E_i)$, and a straightforward calculation shows that the eigenvalues $E_i^{(F=0)}$ can be used to obtain the unfolded spacings

$$s_i = \frac{E_{i+1}-E_i}{E_{i+1}^{(F=0)}-E_i^{(F=0)}}. \tag{10}$$

To recover the unfolded energy levels ϵ_i from the spacing sequence, we can assume that $\epsilon_0=0$.

Statistics

The statistical theory of quantum spectra provides several types of statistics that have been found useful to characterize level fluctuations. Their behavior for integrable or chaotic systems is well known from the appropriate theoretical models [11]. Among all of them we consider the following.

(i) The nearest-neighbor spacing distribution (NNSD) $P(s)$ provides information on the short range correlations among the energy levels of the system. The Poisson distribution $P(s,\beta=0)=\exp(-s)$ is characteristic of integrable systems. On the other hand, the Wigner surmise $P(s,\beta)=a_\beta s^\beta \exp(-b_\beta s^2)$ is almost exact for chaotic systems with $\beta=1$, 2, and 4 [11]. Izrailev's generalization [14]

$$P(s,\beta) = A_\beta s^\beta \exp\left[-\frac{\pi^2\beta}{16}s^2 - \left(B_\beta - \frac{\pi\beta}{4}\right)s\right], \tag{11}$$

where A_β and B_β are constants determined by normalization, seems to be appropriate to extrapolate between $\beta=0$ and the limit $\beta\to\infty$. Indeed, this expression reproduces correctly the Poisson distribution when $\beta=0$ and constitutes a good approximation to the Wigner surmise when $\beta=1$, 2, and 4, and it has been shown [15] that Eq. (11) provides a good description of the NNSD of the Coulomb gas for $0\leqslant\beta\leqslant4$. Moreover, regardless of the value of β, its asymptotic behavior for large s agrees with that obtained from thermodynamic considerations based on Dyson's Coulomb gas [16].

(ii) In order to characterize correlations of different length, we use the δ_n statistic

$$\delta_n = \sum_{i=1}^{n} (s_i - \langle s\rangle) = \epsilon_{n+1} - \epsilon_1 - n, \quad n=1,\ldots,N-1, \tag{12}$$

which represents the deviation of the excitation energy of the $(n+1)$th unfolded level from its average value n. In spite of some peculiarities, the function δ_n has a formal similarity with a time series [5]. Using numerical techniques borrowed from the time series analysis, we can study the long range correlations of the spectra of the $h(F)$ Hamiltonians. One of these methods is the calculation of the power spectrum of the δ_n series, given by

$$P_k^\delta = |\hat{\delta}_k|^2, \tag{13}$$

where $\hat{\delta}_k$ is the Fourier transform of δ_n,

$$\hat{\delta}_k = \frac{1}{\sqrt{N}}\sum_n \delta_n \exp\left(-\frac{2\pi ikn}{N}\right). \tag{14}$$

As shown in [6], RMEs exhibit $1/f$ noise, i.e., P_k^δ follows a power law

$$P_k^\delta = \frac{N}{2\beta\pi^2 k}, \quad \beta=1,2,4, \tag{15}$$

when $k\ll N$ and $N\gg1$. Thus, chaotic quantum systems, characterized by different space-time symmetries and different level repulsions, exhibit the same long range structure of the fluctuations: in all cases the functional dependence of P_k^δ is the same and without any privileged scale.

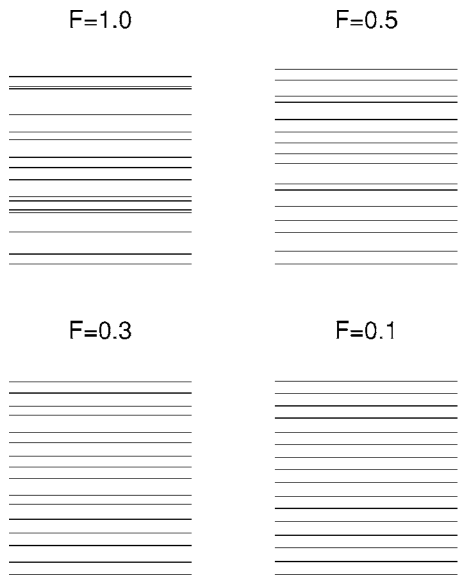

FIG. 1. Short sequences of consecutive energy levels corresponding to four selected tridiagonal matrices $h(F)$ with $F=1$, 0.5, 0.3, and 0.1. In all cases, the average spacing is $\langle s\rangle=1$.

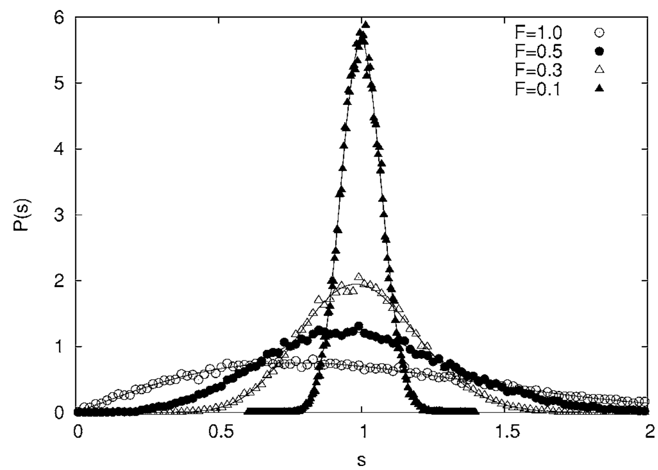

FIG. 2. Numerical values of the NNSD of four RTMEs with $F=1$ (open circles), 0.5 (filled circles), 0.3 (triangles), and 0.1 (filled triangles), compared to the results of a least-squares fit to Eq. (11) (solid lines). The numerical $P(s)$ values are calculated by collecting the spacings of 30 matrices of dimension $N=1000$.

RESULTS AND DISCUSSION

Figure 1 displays short sequences of consecutive unfolded levels corresponding to selected $h(F)$ matrices of dimension $N=1000$ with $F=1$, 0.5, 0.3, and 0.1. Although the four spectra have the same average spacing $\langle s\rangle=1$, their global appearance is different. At $F=1$ there are three pairs of almost degenerate levels. At $F=0.5$ we can still distinguish two pairs of levels lying rather closely, but as F increases the degeneracy disappears gradually and finally the spectrum becomes essentially equidistant at $F=0.1$. This example shows qualitatively the correlation between F and β given by Eq. (6).

In order to make more quantitative this result we have calculated $P(s)$ for RTMEs with the same values of F as in the previous example. To compute $P(s)$, we generate 30 tridiagonal matrices of dimension $N=1000$, diagonalize each matrix and calculate the spacings between consecutive levels, and finally we collect together all the spacings. Figure 2 shows the behavior of $P(s)$ as F increases together with the results of a least-squares fit to Eq. (11). It can be clearly seen how $P(s)$ evolves from a wide and smooth function at $F=1$ to a very picked distribution at $F=0.1$. This result is consistent with the expected picket fence spectrum with $P(s)=\delta(s-1)$ in the $F=0$ limit. The values of β provided by the least-squares fit are shown in Table I. They are in reasonable agreement with the predictions of Eq. (6); moreover, using these values, the four $P(s,\beta)$ curves predicted by Eq. (11) describe perfectly the behavior of the actual $P(s)$ distribution. Therefore, Eq. (11) provides a good description of the short range correlations of the $h(F)$ Hamiltonians; only for large values of level repulsion is β underestimated, as happens in the Dyson's Coulomb gas [15].

However, in this paper we aim at characterizing not only the short range correlations, but the spectral fluctuations of tridiagonal ensembles at all scales. More precisely, the question we are interested in is whether the long range correlation structure (that defines the global appearance of a signal) remains invariant through the whole transition from $F=1$ to $F\to 0$. To answer this question we calculate the power spectrum of the δ_n statistic. Since we already know that the average power spectrum of three classical RMEs follows a power law $\langle P_k^{\delta}\rangle \propto 1/k$, it is worth studying whether RTMEs exhibit the same behavior and if the degree of level repulsion modifies the statistical properties of the whole spectrum or not.

The evolution of the δ_n function can be followed in Fig. 3. Here, we display in different panels the behavior of this statistic for selected matrices with $F=1$, 0.5, 0.3, and 0.1. Note that the axis scales are the same in the four panels. We have also added lower left subpanels where, using the natural scale of each signal, we show small parts of the δ_n series. It is clearly seen that, except for the amplitude of the fluctuations, the correlation structure seems very similar in the four cases; thus, these plots suggest that the correlation structure of δ_n is not dramatically affected by β.

To establish the validity of this preliminary result, we calculate the average power spectrum of δ_n as a function of F and show the result in a doubly logarithmic plot. A twofold average procedure is carried out to reduce the enhancement of the fluctuations caused by these type of plots. First, we generate 30 tridiagonal matrices of dimension $N=1000$, the power spectrum of each matrix is calculated using Eq. (15) and then the ensemble average is performed. Afterwards, the logarithmic frequency axis is divided into equal bins and the power spectrum components are averaged in each bin. The

TABLE I. Dependence of the level repulsion parameter β on the ensemble parameter F.

F	1.0	0.5	0.3	0.1
β	1.0 ± 0.1	3.9 ± 0.1	10.3 ± 0.2	90 ± 1

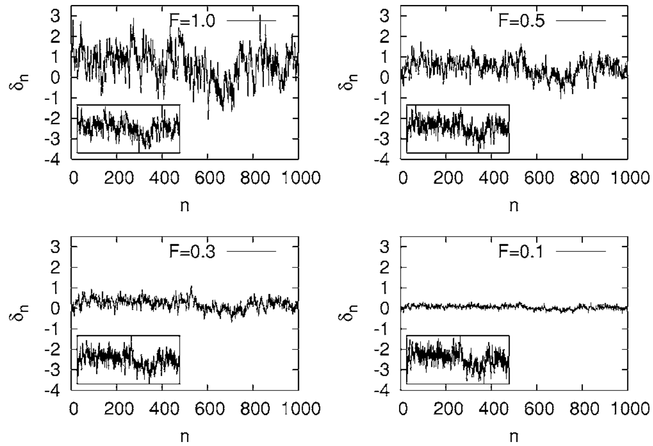

FIG. 3. Comparison of the δ_n statistic for tridiagonal matrices $h(F)$ with $F=1$, 0.5, 0.3, and 0.1.

behavior of $\langle P_k^\delta \rangle$ for ensembles characterized by $F=1$, 0.5, 0.3, and 0.1 is displayed in Fig. 4. Although the amplitude of the fluctuations decreases with F, it is clearly seen that the long range correlation structure is not affected by the increase of the level repulsion: $\langle P_k^\delta \rangle$ shows a neat $1/f$ behavior in all the cases and only the multiplicative factor $N/(2\pi^2\beta)$ of the power spectrum decreases as the repulsion parameter increases. This figure also shows the comparison of the calculated values with the theoretical predictions of Eq. (15) assumed to be valid for all β. Using the values of β quoted in Table I, the agreement between theory and numerics is excellent.

The formal analogy between a energy level spectra and a discrete time series lead to the idea that spectral rigidity is analogous to antipersistence [5]. Using RMEs, atomic nuclei, and quantum billiards [5,6] it was shown that the spectral rigidity of chaotic systems with $\beta=1$, 2, or 4 give rise to spectra that, considered as time series, exhibit $1/f$ noise and therefore are completely antipersistent. This work suggests that this result is also valid for very intense level repulsions, i.e., for $\beta>4$: $1/f$ noise seems to be the characteristic fingerprint of a transition where, preserving its scale-free correlation structure, the spectrum evolves from a fluctuating to a nonfluctuating ordered sequence. Since we have only dealt with tridiagonal matrix ensembles, it is natural to ask whether this conclusion is universal or not; in establishing the validity of our assessment other systems that display a similar ordered and correlated sequence of facts should be studied; it will probably be a difficult task. Nevertheless, RTMEs have several appealing features. When $\beta=1$, 2 or 4 their correlation structure is the same as that of all known chaotic quantum systems in the universal regime of spectral fluctuations. Moreover, if $0\leqslant\beta\leqslant 4$ the short-range correlations of these spectra are very similar to those of Dyson's Coulomb gas (where the distance between neighboring particles plays the role of energy level spacings), since in both cases the $P(s)$ distribution is well described by the Izraeliev distribution. All these results and considerations lead us to assess that any energy spectrum characterized by an intense level repulsion ($\beta\geqslant 1$) exhibits $1/f$ noise, that is, the spectrum is analogous to a fully antipersistent time series; the intensity of the level repulsion, measured by β, does not modify the correlation structure of the time series, but only the amplitude of the fluctuations.

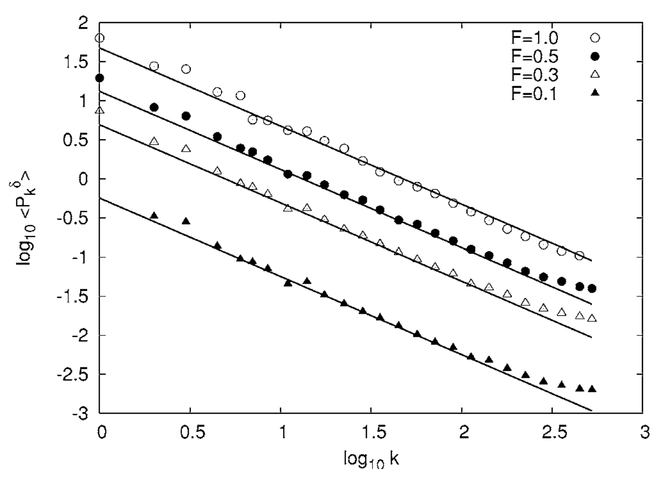

FIG. 4. Average power spectrum P_k^δ for RTMEs with $F=1$ (open circles), 0.5 (filled circles), 0.3 (triangles), and 0.1 (filled triangles), compared to the theoretical predictions of Eq. (15) (solid lines).

SUMMARY AND CONCLUSIONS

Summarizing, we have studied the long range correlation structure of spectra with very high level repulsion.

It was already known that the antipersistence of the time series corresponding to the energy spectrum of a quantum system increases with the level repulsion. The spectra of integrable systems have $\beta=0$ and exhibit $1/f^2$ noise; thus they behave as nonantipersistent series. Chaotic spectra, characterized by $\beta=1$, 2, and 4, exhibit $1/f$ noise and thus, considered as time series, are fully antipersistent. In order to study whether this structure is modified when $\beta>4$, we have generated a family of random matrix ensembles whose members change through a parameter from standard chaotic Hamiltonians with $\beta=1$ to Hamiltonians with picket-fence spectra and $\beta\gg 1$. Using the Lanczos method, the members of a GOE are reduced to tridiagonal form and the amplitude of the tridiagonal elements is modulated by a multiplicative parameter F. As F is modified, the level repulsion parameter charges according to the law $\beta=1/F^2$; when $F=1$ we have $\beta=1$, and as $\beta\to\infty$ as $F\to 0$. Therefore it is possible to study the statistical properties of spectra with any level repulsion intensity.

The ensemble average power spectrum $\langle P_k^\delta \rangle$ has been calculated for representative ensembles with $0\leqslant F\leqslant 1$. A neat power law $\langle P_k^\delta \rangle \propto 1/k$ has been found in all the cases. Therefore, the strong increase of the level repulsion, induced by the suppression of the matrix elements fluctuations, does not modify the long range correlations of these spectra. Consequently, $1/f$ noise arises as the main feature of a transition that reduces to zero fluctuating part of the spectrum (or more generally a signal), preserving its scale-free correlation structure. This is consistent with the idea of antipersistence. Since an antipersistent time series tends to compensate globally any increasing or decreasing trend, it is reasonable to think that in certain limit this tendency is so strong that hinders completely the fluctuations of the signal.

Moreover, due to the fact that the family of tridiagonal Hamiltonian ensembles includes the classic RME, and their members are very similar to a unidimensional Coulomb gas at least if $0 \leqslant \beta \leqslant 4$, we conjecture that any energy spectrum characterized by an intense level repulsion exhibits $1/f$ noise. It is noteworthy commenting here that the Dyson's Coulomb gas can be a good choice to study what happens in a different system with a very similar phenomenology; moreover, the relation between β and the temperature T that characterizes this system may enlighten the conclusions obtained with the tridiagonal matrix ensembles.

ACKNOWLEDGMENT

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